# AN ALGEBRAIC APPROACH TO WEAK AND DELAY BISIMULATION IN COALGEBRA

TOMASZ BRENGOS

Abstract. The aim of this paper is to introduce an algebraic structure on the set of all coalgebras with the same state space over the given type which allows us to present definitions of weak and delay bisimulation for coalgebras. Additionally, we introduce an algebraic structure on the carrier set of the final coalgebra and characterize a special subcoalgebra of the final coalgebra which is used in the formulation of the weak coinduction principle. Finally, the new algebraic setting allows us to present a definition of an approximated weak bisimulation, study its properties and compare it with previously defined weak bisimulation for coalgebras.

## 1. Introduction

The notion of a strong bisimulation for different transition systems plays an important role in theoretical computer science. A weak bisimulation is a relaxation of this notion by allowing silent, unobservable transitions. It is a well established notion for many deterministic and probabilistic transition systems (see [6], [11], [12]). For many state-based systems one can equivalently introduce weak bisimulation in two different ways (see [1] for details).

The notion of a strong bisimulation, unlike the weak bisimulation, has been well captured coalgebraically (see e.g. [4],[15]). Different approaches to defining weak bisimulations for coalgebras have been presented in the literature. The earliest paper is [14], where the author studies weak bisimulations for while programs. In [8] the author introduces a definition of weak bisimulation for coalgebras by translating a coalgebraic structure into an LTS. This construction works for coalgebras over a large class of functors but does not cover the distribution functor, hence it is not applicable to different types of probabilistic systems. In [9] weak bisimulations are introduced via weak homomorphisms. As noted in [13] this construction does not lead to intuitive results for probabilistic systems. In [13] the authors present a definition of weak bisimulation for classes of coalgebras over functors obtained from bifunctors. Here, weak bisimulation of a system is defined as a strong bisimulation of a transformed system. Finally, in [1] we proposed a new approach to defining weak bisimulation in two different ways. Two definitions of weak bisimulation described by us in [1] were proposed in the setting of coalgebras over ordered functors. In particular they lead to proper definitions of weak bisimulation e.g. for labelled transistion systems or simple Segala systems. The key ingredient of the definitions is the notion of a saturator. As noted in [1] the saturator is sometimes too general

*Key words and phrases.* coalgebra, bisimulation, weak bisimulation, delay bisimulation, logic.
This work has been supported by the European Union in the framework of European Social Fund through the Warsaw University of Technology Development Programme and the grant of Warsaw University of Technology no. 504M for young researchers.

to model only weak bisimulation and may be used to define other equivalences, e.g. delay bisimulation (see [1] and Section 2 for details). The aim of this paper is to present a general coalgebraic setting in which it is possible to distinguish between the types of saturators used in coalgebraic definition of weak bisimulation and delay bisimulation by introducing an algebraic structure on coalgebras over the same state space. The new setting also allows us to define weak and delay bisimulation where the attempt from [1] failed to work (see Section 5 for details) namely via approximated weak and approximated delay bisimulation.

The paper is organized as follows. In Section 2 we present basic definitions and facts used in the rest of the paper. Section 3 is devoted to presenting the algebraic structure on the set of all coalgebras over ordered functors (here, unlike in [1], we additionally assume the order to be complete) with a common state space and show its basic properties. We show how to define saturators for weak and delay bisimulation for coalgebras in terms of the new algebraic operations. We support our general coalgebraic setting by presenting some examples which involve labelled transition systems and simple Segala systems. In Section 4 we recall some facts regarding weak coinduction principle stated in [1] and present a definition of operators on the set of states of the final coalgebra which give us an axiomatization of the set of all weakly saturated states. The set of all weakly saturated states is used in formulation of weak coinduction principle. Finally, Section 5 is devoted to defining approximated weak bisimulation and approximated delay bisimulation via saturator approximants and giving the sufficient conditions for equivalence of the new definition with the standard weak (delay) bisimulation defined in this paper.

## 2. Basic notions and properties

Let $\mathsf{Set}$ be the category of all sets and mappings between them. Let $F\colon \mathsf{Set} \to \mathsf{Set}$ be a functor. An *$F$-coalgebra* is a tuple $\langle A, \alpha\rangle$, where $A$ is a set and $\alpha$ is a mapping $\alpha\colon A \to FA$. The set $A$ is called a *carrier* and the mapping $\alpha$ is called a *structure* of the coalgebra $\langle A, \alpha\rangle$.

A *homomorphism* from an $F$-coalgebra $\langle A, \alpha\rangle$ to a $F$-coalgebra $\langle B, \beta\rangle$ is a mapping $f\colon A \to B$ such that $T(f) \circ \alpha = \beta \circ f$.

An $F$-coalgebra $\langle S, \sigma\rangle$ is said to be a *subcoalgebra* of an $F$-coalgebra $\langle A, \alpha\rangle$ whenever there is an injective homomorphism from $\langle S, \sigma\rangle$ into $\langle A, \alpha\rangle$. This fact is denoted by $\langle S, \sigma\rangle \leq \langle A, \alpha\rangle$.

Let $\mathsf{CSLat}$ be the category of all join complete semilattices and mappings that preserve any suprema. Note that there is a forgetful functor $U : \mathsf{CSLat} \to \mathsf{Set}$ assigning to each poset $(X, \leq)$ the underlying set $X$ and to each monotonic map $f : (X, \leq) \to (Y, \leq)$ the map $f : X \to Y$.

In this paper we focus our attention only on functors $F : \mathsf{Set} \to \mathsf{CSLat}$. We may naturally assign to $F$ its composition $\bar{F} = U \circ F$ with the forgetful functor $U : \mathsf{CSLat} \to \mathsf{Set}$. For sake of simplicity of notation most of the times we will identify the functor $F : \mathsf{Set} \to \mathsf{CSLat}$ with the $\mathsf{Set}$-endofunctor $\bar{F} = U \circ F$ and write $F$ to denote both $F$ and $\bar{F}$.

**Remark 2.1.** Note that in [1] we considered more general functors, i.e. $F : \mathsf{Set} \to \mathsf{Pos}$, where $\mathsf{Pos}$ denotes the category of all posets as objects and monotonic maps between them as morphisms. In this paper we require from the order on $FX$ to be join complete (hence, complete) and that for any $f : X \to Y$ the mapping $Ff : FX \to FY$ preserves any suprema.

**Example 2.2.** The powerset functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ may be regarded as $\mathcal{P} : \mathsf{Set} \to \mathsf{CSLat}$, where for any set $X$ the set $\mathcal{P}X$ is ordered by inclusion.

**Example 2.3.** Note that for any functor $H : \mathsf{Set} \to \mathsf{Set}$ the composition $\mathcal{P}H$ may be regarded as $\mathcal{P}H : \mathsf{Set} \to \mathsf{CSLat}$ with a natural ordering given by inclusion. In this paper we will focus our attention on coalgebras over the following functors:

- $\mathcal{P}(\Sigma \times \mathcal{I}d)$,
- $\mathcal{P}(\Sigma \times \mathcal{D})$,

where $\mathcal{D}$ is the distribution functor, i.e. a functor which assigns to any set $X$ the set $\mathcal{D}X := \{\mu : X \to [0,1] \mid \sum_{x\in X} \mu(x) = 1\}$ of discrete measures and to any mapping $f : X \to Y$ a mapping $\mathcal{D}f : \mathcal{D}X \to \mathcal{D}Y; \mu \mapsto \mathcal{D}f(\mu) : Y \to [0,1]; y \mapsto \sum_{f(x)=y} \mu(x)$. The coalgebras over the first functor are exactly labelled transition systems. The $\mathcal{P}(\Sigma \times \mathcal{D})$-coalgebras are simple Segala systems introduced and thoroughly studied in [11],[12].

Note that if we consider any functor $F : \mathsf{Set} \to \mathsf{CSLat}$ then we may introduce for any $X, Y \in \mathsf{Set}$ a join complete semilattice structure on the set $Hom(X, FY)$ as follows. For a family of maps $\{f_i\}_{i\in I} \subseteq Hom(X, FY)$ define their supremum $\sum_{i\in I} f_i$ as follows:

$$\sum_{i\in I} f_i : X \to FY; x \mapsto \sum_{i\in I} f_i(x).$$

Given $f : X \to Y$, $\alpha : X \to FZ$, $g : Z \to U$ and $\beta : Y \to FU$ an inequality $Fg\circ\alpha \leq \beta\circ f$ will be denoted by a diagram on the left and an equality $Fg\circ\alpha = \beta\circ f$ will be denoted by a diagram on the right:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\leq"] & Y \arrow[d, "\beta"] \\ FZ \arrow[r, "Fg"'] & FU \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "="] & Y \arrow[d, "\beta"] \\ FZ \arrow[r, "Fg"'] & FU \end{tikzcd}$$

We will now recall some basic definitions and properties from [1].

**Definition 2.4.** Let $U : \mathsf{Set}_F \to \mathsf{Set}$ be the forgetful functor. A *coalgebraic operator* $\mathfrak{o}$ is a functor $\mathfrak{o} : \mathsf{Set}_F \to \mathsf{Set}_F$ such that the following diagram commutes:

$$\begin{tikzcd} \mathsf{Set}_F \arrow[r, "\mathfrak{o}"] \arrow[dr, "U"'] & \mathsf{Set}_F \arrow[d, "U"] \\ & \mathsf{Set} \end{tikzcd}$$

In other words, if $f : A \to B$ is a homomorphism between two $F$-coalgebras $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ then $f$ is a homomorphism between $\langle A, \mathfrak{o}\alpha\rangle$ and $\langle B, \mathfrak{o}\beta\rangle$, i.e.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "="] & B \arrow[d, "\beta"] \\ FB \arrow[r, "Ff"'] & FB \end{tikzcd} \implies \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathfrak{o}\alpha"'] \arrow[dr, phantom, "="] & B \arrow[d, "\mathfrak{o}\beta"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd}$$

In [1] we introduced a notion of a saturator. We will recall this definition now. We say that a coalgebraic operator $\mathfrak{s}$ with respect to a class $\mathsf{C}$ is a *saturator* if for any two $F$-coalgebras $\langle A, \alpha\rangle$, $\langle B, \beta\rangle$ from $\mathsf{C}$ and any mapping $f : A \to B$ the

inequality $Ff \circ \alpha \leq \mathfrak{s}\beta \circ f$ is equivalent to $Ff \circ \mathfrak{s}\alpha \leq \mathfrak{s}\beta \circ f$. We may express the property in diagrams as follows:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\leq"] & B \arrow[d, "\mathfrak{s}\beta"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd} \iff \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathfrak{s}\alpha"'] \arrow[dr, phantom, "\leq"] & B \arrow[d, "\mathfrak{s}\beta"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd}$$

**Lemma 2.5.** [1] *Let $\mathfrak{s} : \mathsf{C} \to \mathsf{Set}_F$ be an operator w.r.t. a full subcategory $\mathsf{C}$ of $\mathsf{Set}_F$ and additionally let $\mathfrak{s}(\mathsf{C}) \subseteq \mathsf{C}$. Then $\mathfrak{s}$ is a saturator if and only if it satisfies the following three properties:*

- *$\alpha \leq \mathfrak{s}\alpha$ for any coalgebra $\langle A, \alpha \rangle \in \mathsf{C}$ (extensivity),*
- *$\mathfrak{s} \circ \mathfrak{s} = \mathfrak{s}$ (idempotency),*
- *if $Ff \circ \alpha \leq \beta \circ f$ then $Ff \circ \mathfrak{s}\alpha \leq \mathfrak{s}\beta \circ f$ for any $f : X \to Y$ (monotonicity):*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\leq"] & B \arrow[d, "\beta"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd} \implies \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathfrak{s}\alpha"'] \arrow[dr, phantom, "\leq"] & B \arrow[d, "\mathfrak{s}\beta"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd}$$

**Remark 2.6.** In this paper we will only focus on saturators for which $\mathsf{C} = \mathsf{Set}_F$. In other words, here we define a saturator as a coalgebraic operator satisfying the properties from Lemma 2.5.

**Example 2.7.** The identity coalgebraic operator $\mathfrak{id} : \mathsf{Set}_F \to \mathsf{Set}_F$ is a trivial example of a saturator.

**Example 2.8.** Let $\tau \in \Sigma$ be a silent transition label. For a coalgebra structure $\alpha : A \to \mathcal{P}(\Sigma \times A)$ we define its saturation $\mathfrak{s}\alpha : A \to \mathcal{P}(\Sigma \times A)$ as follows. For any element $a \in A$ put

$$\mathfrak{s}\alpha(a) := \alpha(a) \cup \{(\tau, a') \mid a \stackrel{\tau^*}{\to} a'\} \cup \{(\sigma, a') \mid a \stackrel{\tau^*}{\to} \circ \stackrel{\sigma}{\to} \circ \stackrel{\tau^*}{\to} a' \text{ for } \sigma \neq \tau\}.$$

**Example 2.9.** Let $\tau \in \Sigma$ be a silent transition label. For a coalgebra structure $\alpha : A \to \mathcal{P}(\Sigma \times A)$ we define its saturation $\mathfrak{s}\alpha : A \to \mathcal{P}(\Sigma \times A)$ as follows. For any element $a \in A$ put

$$\mathfrak{s}\alpha = \{(\tau, a') \mid a \stackrel{\tau^*}{\to} a'\} \cup \{(\sigma, a') \mid a \stackrel{\tau^*}{\to} \circ \stackrel{\sigma}{\to} a' \text{ for } \sigma \neq \tau\}.$$

This saturation is associated with the so-called delay bisimulation for LTS. For more information the reader is referred to [7] or Section 3.

**Example 2.10.** For the functor $F = \mathcal{P}(\Sigma \times \mathcal{D})$, an $F$-coalgebra $\langle A, \alpha \rangle$, a state $a \in A$ and $\sigma \in \Sigma$ we write $a \stackrel{\sigma}{\to}_\alpha \mu$ if $(\sigma, \mu) \in \alpha(a)$. For a state $a \in A$ and a measure $\nu \in \mathcal{D}(\Sigma \times A)$ a pair $(a, \nu)$ is called a *step* in $\langle A, \alpha \rangle$ if there is $\sigma \in A$ and $\mu \in \mathcal{D}A$ such that $a \stackrel{\sigma}{\to}_\alpha \mu$ and $\nu(\sigma, a') = \mu(a')$ for any $a' \in A$. A *combined step* in $\langle A, \alpha \rangle$ is a pair $(a, \nu)$, where $a \in A$ and $\nu \in \mathcal{D}(\Sigma \times A)$ for which there is a countable family of positive numbers $\{p_i\}_{i \in I}$ such that $\sum_{i \in I} p_i = 1$ and a countable family of steps $\{(a, \nu_i)\}_{i \in I}$ in $\langle A, \alpha \rangle$ such that $\nu = \sum_{i \in I} p_i \cdot \nu_i$. The definition of a combined step is a slight modification of a similar definition presented in [11]. The notion of weak arrows $\stackrel{\sigma}{\Longrightarrow}_P$ remains the same regardless of the small difference between the two definitions. For any $a \in A$ let $\delta_a \in \mathcal{D}A$ denote the distribution for which $\delta_a(a) = 1$. Let $\tau \in \Sigma$ be the invisible transition. As in [11] for any $\sigma \in \Sigma$ we write $a \stackrel{\sigma}{\Longrightarrow}_P \mu$ whenever $\sigma = \tau$ and $\mu = \delta_a$ or there is a combined step $(a, \nu)$ in $\langle A, \alpha \rangle$ such that

if $(\sigma', a') \notin \{\sigma, \tau\} \times A$ then $\nu(\sigma', a') = 0$ and $\mu = \sum_{(\sigma',a') \in \{\sigma,\tau\} \times A} \nu(\sigma', a') \cdot \mu_{(\sigma',a')}$ and if $\sigma' = \sigma$ then $a' \overset{\tau}{\Longrightarrow}_P \mu_{(\sigma',a')}$ otherwise $\sigma' = \tau$ and $a' \overset{\sigma}{\Longrightarrow}_P \mu_{(\sigma',a')}$. Note that the family of sets $\{\mu \in \mathcal{D}A \mid a \overset{\sigma}{\Longrightarrow}_P \mu\}$ indexed by $(\sigma, a) \in \Sigma \times A$ is the smallest possible satisfying the above conditions. Now, define $\mathfrak{s}\alpha : A \to FA$ by putting $\mathfrak{s}\alpha(a) := \{(\sigma, \mu) \mid a \overset{\sigma}{\Longrightarrow}_P \mu\}$ for any $a \in A$.

We will now recall two approaches to defining weak bisimulations proposed in [1]. Since the notions depend on a saturator $\mathfrak{s}$ and, as will be noted in examples to follow, encompass more than weak bisimulation we will use the term $\mathfrak{s}$-bisimulation ($\mathfrak{s}$-saturated bisimulation) instead of weak bisimulation (resp. saturated weak bisimulation).

**Definition 2.11.** A relation $R \subseteq A \times B$ is called an $\mathfrak{s}$-*bisimulation* between $F$-coalgebras $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ provided that there is a structure $\gamma_1 : R \to FR$ and a structure $\gamma_2 : R \to FR$ for which:

- $\alpha \circ \pi_1 = F\pi_1 \circ \gamma_1$ and $F\pi_2 \circ \gamma_1 \leq \mathfrak{s}\beta \circ \pi_2$,
- $\beta \circ \pi_2 = F\pi_2 \circ \gamma_2$ and $F\pi_1 \circ \gamma_2 \leq \mathfrak{s}\alpha \circ \pi_1$.

$$\begin{tikzcd} A \arrow[d, "\alpha"'] \arrow[dr, phantom, "="] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma_1"] \arrow[dr, phantom, "\leq"] & B \arrow[d, "\mathfrak{s}\beta"] \\ FA & FR \arrow[l, "F\pi_1"] \arrow[r, "F\pi_2"'] & FB \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "\mathfrak{s}\alpha"'] \arrow[dr, phantom, "\geq"] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma_2"] \arrow[dr, phantom, "="] & B \arrow[d, "\beta"] \\ FA & FR \arrow[l, "F\pi_1"] \arrow[r, "F\pi_2"'] & FB \end{tikzcd}$$

**Definition 2.12.** A relation $R \subseteq A \times B$ is said to be a $\mathfrak{s}$-*saturated bisimulation* between $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ provided that there is a structure $\gamma : R \to FR$ for which the following diagram commutes:

$$\begin{tikzcd} A \arrow[d, "\mathfrak{s}\alpha"'] \arrow[dr, phantom, "="] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] \arrow[dr, phantom, "="] & B \arrow[d, "\mathfrak{s}\beta"] \\ FA & FR \arrow[l, "F\pi_1"] \arrow[r, "F\pi_2"'] & FB \end{tikzcd}$$

For a saturator $\mathfrak{s}$ and an $F$-coalgebra $\langle A, \alpha \rangle$ let $\approx_{\mathfrak{s}}$ ($\approx_{\mathfrak{s}}^{sat}$) denote the largest $\mathfrak{s}$-bisimulation (resp. $\mathfrak{s}$-saturated bisimulation) on $\langle A, \alpha \rangle$. Such relations always exist and are equivalence relations provided the functor $F$ considered as a $\mathsf{Set}$-endofunctor weakly preserves pullbacks (see [1] for details). We say that a functor $F : \mathsf{Set} \to \mathsf{CSLat}$ *preserves downsets* provided that for any $f : X \to Y$ and any $\vec{x} \in FX$ the following equality holds:

$$Ff(\vec{x}\downarrow) = Ff(\{\vec{x}' \in FX \mid \vec{x}' \leq \vec{x}\}) = Ff(\vec{x})\downarrow = \{\vec{y} \in FY \mid \vec{y} \leq Ff(\vec{x})\}.$$

**Theorem 2.13.** [1] *Let $F : \mathsf{Set} \to \mathsf{CSLat}$ preserve downsets and let $F$ considered as a $\mathsf{Set}$-endofunctor weakly preserve pullbacks. Then for any $F$-coalgebra $\langle A, \alpha \rangle$ we have*

$$\approx_{\mathfrak{s}} = \approx_{\mathfrak{s}}^{sat}.$$

The above theorem tells us that putting rather mild restrictions on $F$ two states in an $F$-coalgebra $\langle A, \alpha \rangle$ are $\mathfrak{s}$-bisimilar if and only if they are $\mathfrak{s}$-saturated bisimilar. Hence, from now on we will only use the former definition. The latter one has a major drawback. To show that two states $a, b$ in $\langle A, \alpha \rangle$ are $\mathfrak{s}$-saturated bisimilar we need to know $\mathfrak{s}\alpha(a)$ and $\mathfrak{s}\alpha(b)$. This is not neccessary when showing they are $\mathfrak{s}$-bisimilar. Indeed, sometimes it is enough to know $\alpha(a)$, $\alpha(b)$ and an approximation from below of $\mathfrak{s}\alpha(a)$ and $\mathfrak{s}\alpha(b)$. Therefore, from the point of view of automated

reasoning and computation it is easier to prove two stated are $\mathfrak{s}$-bisimilar. Moreover, in Section 5 we show that for some coalgebras (e.g. locally finite labelled transition systems) the knowledge of the saturated structure $\mathfrak{s}\alpha$ is not neccessary at all as it is enough to consider a set of its approximants.

**Example 2.14.** For labelled transition system coalgebras and a saturator defined in Example 2.8 the $\mathfrak{s}$-bisimulation is a weak bisimulation for LTS [6]. For the saturator from Example 2.9 the $\mathfrak{s}$-bisimulation is a delay bisimulation for LTS (see Section 3 for details). Moreover, the $\mathfrak{id}$-bisimulation for the saturator from Example 2.7 for LTS coincides with the notion of a strong bisimulation.

**Example 2.15.** For simple Segala system coalgebras, i.e. coalgebras of the type $\mathcal{P}(\Sigma \times \mathcal{D})$ and saturator from Example 2.10 the $\mathfrak{s}$-bisimulation is the weak probabilistic bisimulation defined in [12].

In Example 2.14 we see that the notion of a saturator and $\mathfrak{s}$-bisimulation introduced in [1] encompasses more than neccessary to introduce weak bisimulations coalgebraically. This means that the theory of saturators should be equipped with tools to be able to differentiate between types of saturators.

## 3. Weak and delay bisimulation saturators

They aim of this section is to present an algebraic setting thanks to which it is be possible to distinguish between different types of saturators used in the definition of $\mathfrak{s}$-bisimulation. The main idea behind the definition of weak and delay bisimulation and their saturators is that we allow multiple silent steps to take place. The difference between weak and delay bisimulation is that in delay bisimulation we only allow them to take place a visible transition occurs, whereas in weak bisimulation they may also take place after the visible step (see [10] for details). Given a labelled transition system coalgebra $\langle A, \alpha\rangle$ and the weak bisimulation saturator $\mathfrak{s}$ from Example 3.11 the saturated structure $\mathfrak{s}\alpha$ can be defined using the following four inference rules.

$$(1)\frac{x \stackrel{\sigma}{\rightarrow}_\alpha x}{x \stackrel{\sigma}{\rightarrow}_{\mathfrak{s}\alpha} x} \quad (2)\frac{}{x \stackrel{\tau}{\rightarrow}_{\mathfrak{s}\alpha} x} \quad (3)\frac{x \stackrel{a}{\rightarrow}_\alpha y \qquad y \stackrel{\tau}{\rightarrow}_\alpha z}{x \stackrel{a}{\rightarrow}_{\mathfrak{s}\alpha} z}$$

$$(4)\frac{x \stackrel{\tau}{\rightarrow}_\alpha y \qquad y \stackrel{a}{\rightarrow}_\alpha z}{x \stackrel{a}{\rightarrow}_{\mathfrak{s}\alpha} z}$$

Moreover, for the delay saturator $\mathfrak{s}$ from Example 2.9 and the structure $\mathfrak{s}\alpha$ is defined as a structure which satisfies rules (1), (2) and (4). Therefore, to define and distinguish weak and delay bisimulation saturators for coalgebras it is enough to be able to speak about a silent step loop which transitions from a state to itself and to able to compose the silent part of a structure with the whole structure (from the left and from the right) on an abstract coalgebraic level.

As noted in Section 2 the set of all coalgebraic structures $Hom(A, FA)$ over a state space $A$ is endowed with a complete join semilattice structure. Let $0_A$ denote the smallest element in $Hom(A, FA)$, i.e. $0_A = \sum \varnothing$. In the general setting the definition of a saturated structure $\alpha^w$ for weak bisimulation or $\alpha^d$ for delay bisimulation will be captured using abstract algebraic operations defined on the set of all coalgebras with a common carrier. Let $A$ be a set. From now on we assume we are given two binary operations

$$\triangleleft, \triangleright : Hom(A, FA)^2 \to Hom(A, FA)$$

and a constant

$$1_A \in Hom(A, FA).$$

For sake of simplicity of notation we will drop the subscript and write 1 and 0 instead of $1_A$ and $0_A$ whenever the carrier set can be deduced from the context. The two binary operations and the constants $0, 1$ are assumed to satisfy the following list of properties.

(a) $(\alpha \triangleleft 1) \triangleleft \alpha' = \alpha \triangleleft \alpha'$ and $\alpha' \triangleright (\alpha \triangleleft 1) = \alpha' \triangleright \alpha$,
(b) $(\alpha \triangleleft 1) \triangleright \alpha' = (\alpha \triangleright \alpha') \triangleleft 1$ and $\alpha \triangleleft (\alpha' \triangleleft 1) = (\alpha \triangleleft \alpha') \triangleleft 1$,
(c) $1 \triangleleft \alpha \geq \alpha$, $\alpha \triangleleft 1 \leq \alpha$ and $\alpha \triangleright 1 = \alpha$,
(d) $(\alpha \triangleleft 1) \triangleleft (\alpha' \triangleleft 1) = (\alpha \triangleleft 1) \triangleright (\alpha' \triangleleft 1)$ and $\left(\sum_{i\in I} \alpha_i\right) \triangleleft 1 = \sum_{i\in I} (\alpha_i \triangleleft 1)$,
(e) $\alpha \diamond 0 = 0 \diamond \alpha = 0$ for $\diamond \in \{\triangleleft, \triangleright\}$,
(f) any mapping $f : A \to B$ is a homomorphism between $1_A : A \to FA$ and $1_B : B \to FB$. The same applies to $0_A$ and $0_B$,
(g) if a mapping $f : A \to B$ is a coalgebraic homomorphism between coalgebras $\alpha : A \to FA$ and $\beta : B \to FB$, $\alpha' : A \to FA$ and $\beta' : B \to FB$ then $f$ is a coalgebraic homomorphism between $\alpha \diamond \alpha'$ and $\beta \diamond \beta'$ for $\diamond \in \{\triangleleft, \triangleright\}$,
(h) if for coalgebraic structures $\alpha, \alpha' : A \to FA$, $\beta, \beta' : B \to FB$ a mapping $f : A \to B$ satisfies

$$Ff \circ \alpha \leq \beta \circ f \text{ and } Ff \circ \alpha' \leq \beta' \circ f$$

then

$$Ff \circ (\alpha \diamond \alpha') \leq (\beta \diamond \beta') \circ f \text{ for } \diamond \in \{\triangleleft, \triangleright\}.$$

**Remark 3.1.** For two coalgebraic structures $\alpha, \beta : A \to FA$ the structure $\alpha \triangleleft \beta$ and $\alpha \triangleright \beta$ should be interpreted as an abstract composition of the structure $\beta$ together with the silent, unobservable part of the structure $\alpha$ (resp. silent part of $\beta$ with full structure $\alpha$). The structure $1_A : A \to FA$ for any set $A$ should be thought of as a step which for any state $a \in A$ transitions back to itself using the silent label. The structure 0 can be thought of as a no-transition structure. Given a coalgebraic structure $\alpha : A \to FA$ the structure $\alpha \triangleleft 1$ should be interpreted as part of $\alpha$ which contains the silent transitions only. Moreover, as we will see further on, by Lemma 3.3 the operator $(-) \triangleleft 1$ may be thought of as being dual to a saturator. The first equation from (a) says that if one first extracts all silent steps from $\alpha$ by calculating $\alpha \triangleleft 1$ and compose all steps from $\alpha'$ with the silent part of $\alpha \triangleleft 1$ then it is the same as composing $\alpha'$ with silent part of $\alpha$. The second equality in (a) is a dual equality. The first equality from (b) describes what happens if we compose the silent part of $\alpha'$ with the full structure of the silent part of $\alpha$. The 2nd equality is dual to the first one. Rule (c) describes the properties of $\triangleleft$- and $\triangleright$-composing a structure $\alpha$ with 1. Property (d) states that both compositions $\triangleleft$ and $\triangleright$ coincide when arguments are only silent transitions and that a join of silent parts of a family of structures is a silent part of the join of these structures. Property (e) tells us what happens if we compose the structure 0 with any other structure. Finally, (f)-(h) are self-explanatory.

**Remark 3.2.** Note that for a coalgebra $\langle A, 1_A\rangle$ any element $a \in A$ forms a one-element subcoalgebra $\langle \{a\}, 1_{\{a\}}\rangle$. This fact follows directly from (f). Similar statement is true for $\langle A, 0_A\rangle$.

**Lemma 3.3.** *Let $\alpha : A \to FA$ be a structure. Then:*

- $1 \triangleleft 1 = 1 \triangleright 1 = 1$,

- *$\alpha \lhd 1 \leq \alpha$,*
- *$(\alpha \lhd 1) \lhd 1 = \alpha \lhd 1$,*
- *for any $f : A \to B$ the following implications hold:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "="] & B \arrow[d, "\beta"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd} \Longrightarrow \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha\lhd 1"'] \arrow[dr, phantom, "="] & B \arrow[d, "\beta\lhd 1"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd}$$

*and*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\leq"] & B \arrow[d, "\beta"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd} \Longrightarrow \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha\lhd 1"'] \arrow[dr, phantom, "\leq"] & B \arrow[d, "\beta\lhd 1"] \\ FA \arrow[r, "Ff"'] & FB \end{tikzcd}$$

*Proof.* The first, second and third assertion follow directly from (a), (c) and (d). The last statement is a direct consequence of (f),(g) and (h). □

**Example 3.4.** For labelled transition system coalgebras of the type $\mathcal{P}(\Sigma \times \mathcal{I}d)$ put

$$1_A : A \to \mathcal{P}(\Sigma \times A); a \mapsto (\tau, a).$$

For coalgebras $\alpha, \alpha' : A \to \mathcal{P}(\Sigma \times A)$ over the same set of states $A$ we define $\alpha' \lhd \alpha : A \to \mathcal{P}(\Sigma \times A)$ as follows. For a state $a \in A$, letter $\sigma \in \Sigma$ we have $a \xrightarrow{\sigma}_{\alpha' \lhd \alpha} a'$ if there is $a'' \in A$ for which $a \xrightarrow{\sigma}_{\alpha} a''$ and $a'' \xrightarrow{\tau}_{\alpha'} a'$. Conversely, we put $a \xrightarrow{\sigma}_{\alpha' \rhd \alpha} a'$ if there is $a'' \in A$ for which $a \xrightarrow{\tau}_{\alpha} a''$ and $a'' \xrightarrow{\sigma}_{\alpha'} a'$. The operations $1, \lhd, \rhd$ satisfy properties (a)-(h).

**Example 3.5.** For simple Segala systems, i.e. coalgebras of the type $\mathcal{P}(\Sigma \times \mathcal{D})$ for any set $A$ put

$$1_A : A \to \mathcal{P}(\Sigma \times \mathcal{D}A); a \mapsto \delta_a.$$

Now for coalgebras $\alpha, \alpha' : A \to \mathcal{P}(\Sigma \times \mathcal{D}A)$ over the same set of states $A$ we define $\alpha' \lhd \alpha : A \to \mathcal{P}(\Sigma \times \mathcal{D}A)$ as follows. For a state $a \in A$, letter $\sigma \in \Sigma$ and measure $\mu \in \mathcal{D}A$ we have $a \xrightarrow{\sigma}_{\alpha' \lhd \alpha} \mu$ if there is a combined step $(a, \nu)$ in $\langle A, \alpha \rangle$ such that $\nu(\sigma', a') = 0$ for any $a' \in A$ and $\sigma' \neq \sigma$ and $\mu = \sum_{a' \in A} \nu(\sigma, a') \cdot \mu_{a'}$, where $\mu_{a'} \in \mathcal{D}A$ such that $a' \xrightarrow{\tau}_{\alpha'} \mu_{a'}$. We put $a \xrightarrow{\sigma}_{\alpha' \rhd \alpha} \mu$ if there is a combined step $(a, \nu)$ in $\langle A, \alpha \rangle$ such that $\nu(\sigma', a') = 0$ for any $a' \in A$ and $\sigma' \neq \tau$ and $\mu = \sum_{a' \in A} \nu(\tau, a') \cdot \mu_{a'}$, where $\mu_{a'} \in \mathcal{D}A$ such that $a' \xrightarrow{\sigma}_{\alpha'} \mu_{a'}$. The proof that $\lhd, \rhd$, 0 and 1 satisfy the desired properties (a)-(h) can be found in Appendix.

**Example 3.6.** For simple Segala systems coalgebras we may define the operations $\lhd$, $\rhd$ and 1 differently. For any set $A$ put

$$1_A : A \to \mathcal{P}(\Sigma \times \mathcal{D}A); a \mapsto \delta_a.$$

For coalgebras $\alpha, \alpha' : A \to \mathcal{P}(\Sigma \times \mathcal{D}A)$ over the same set of states $A$ we define $\alpha' \lhd \alpha : A \to \mathcal{P}(\Sigma \times \mathcal{D}A)$ as follows. For a state $a \in A$, letter $\sigma \in \Sigma$ and measure $\mu \in \mathcal{D}A$ we have $a \xrightarrow{\sigma}_{\alpha' \lhd \alpha} \mu$ if there is a measure $\mu' \in \mathcal{D}A$ such that $a \xrightarrow{\sigma} \mu'$ and $\mu = \sum_{a' \in A} \mu'(a') \cdot \mu_{a'}$, where $\mu_{a'} \in \mathcal{D}A$ such that $a' \xrightarrow{\tau}_{\alpha'} \mu_{a'}$. Moreover, we put $a \xrightarrow{\sigma}_{\alpha' \rhd \alpha} \mu$ if there is a measure $\mu' \in \mathcal{D}A$ with $a \xrightarrow{\tau} \mu'$ and $\mu = \sum_{a' \in A} \mu'(a') \cdot \mu_{a'}$, where $\mu_{a'} \in \mathcal{D}A$ such that $a' \xrightarrow{\sigma}_{\alpha'} \mu_{a'}$. The proof that the operations $\lhd$, $\rhd$ and 1 satisfy the properties (a)-(h) is a simplification of the analogous proof for Example 3.5 in Appendix and therefore is omitted.

Now consider a family $\{W_A\}_{A\in\mathsf{Set}}$ of mappings

$$W_A : Hom(A, FA) \to Hom(A, FA); \alpha \mapsto 1 + \alpha + \alpha \triangleleft \alpha + \alpha \triangleright \alpha.$$

For sake of simplicity of notation in most of the cases we will drop the index and write $W$ instead of $W_A$.

**Lemma 3.7.** *For any $\alpha : A \to FA$ and $\beta : B \to FB$ we have*

*(1)* $\alpha \le W(\alpha)$,
*(2) if $f : A \to B$ is a homomorphism between $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ then it is a homomorphism between $\langle A, W(\alpha)\rangle$ and $\langle B, W(\beta)\rangle$,*
*(3) for a mapping $f : A \to B$ we have*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & \le & \downarrow{\scriptstyle\beta} \\ FA & \xrightarrow[Ff]{} & FB \end{array} \implies \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle W\alpha}\downarrow & \le & \downarrow{\scriptstyle W\beta} \\ FA & \xrightarrow[Ff]{} & FB \end{array}$$

*Proof.* We will only prove (3) since assertion (2) is proved in a similar manner and (1) is obvious. Assume $Ff \circ \alpha \le \beta \circ f$. We have

$$Ff \circ W\alpha = Ff \circ (1 + \alpha + \alpha \triangleleft \alpha + \alpha \triangleright \alpha) =$$
$$Ff \circ 1 + Ff \circ (\alpha \triangleleft \alpha) + Ff \circ (\alpha \triangleright \alpha) \overset{\text{(f)\&(h)}}{\le}$$
$$1 \circ f + (\beta \triangleleft \beta) \circ f + (\beta \triangleright \beta) \circ f = (1 + \beta + \beta \triangleleft \beta + \beta \triangleright \beta) \circ f.$$

□

We define $W^\lambda\alpha$ for any $\lambda \in \mathsf{Ord}$ by a transfinite induction:

$$W^\lambda\alpha = \begin{cases} W(W^{\lambda'}(\alpha)) & \text{if } \lambda \text{ is a successor ordinal and } \lambda = \lambda' + 1, \\ \sum_{\lambda'<\lambda} W^{\lambda'}\alpha & \text{if } \lambda \text{ is a limit ordinal.} \end{cases}$$

**Lemma 3.8.** *For any $\alpha : A \to FA$, $\beta : B \to FB$ and any ordinal number $\lambda \in \mathsf{Ord}$ we have*

*(1) if $f : A \to B$ is a homomorphism between $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ then it is a homomorphism between $\langle A, W^\lambda(\alpha)\rangle$ and $\langle B, W^\lambda(\beta)\rangle$,*
*(2) for a mapping $f : A \to B$ we have*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & \le & \downarrow{\scriptstyle\beta} \\ FA & \xrightarrow[Ff]{} & FB \end{array} \implies \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle W^\lambda\alpha}\downarrow & \le & \downarrow{\scriptstyle W^\lambda\beta} \\ FA & \xrightarrow[Ff]{} & FB \end{array}$$

*Proof.* We will only prove assertion (2) since (1) is true by a similar argument applying transfinite induction. If $\lambda = 1$ then the statement is true by Lemma 3.7. Let $\lambda \in \mathsf{Ord}$ be any ordinal number greater than 1 and let us assume the statement is true for any ordinal $\lambda' < \lambda$. If $\lambda$ is a successor ordinal such that for $\lambda' \in \mathsf{Ord}$ we have $\lambda = \lambda' + 1$ then since $Ff \circ W^{\lambda'}\alpha \le f \circ W^{\lambda'}\beta$ by Lemma 3.7 we get

$$Ff \circ W^\lambda\alpha = Ff \circ W(W^{\lambda'}\alpha) \le W(W^{\lambda'}\beta) \circ f = W^\lambda\beta \circ f.$$

If $\lambda$ is a limit ordinal then $W^\lambda\alpha = \sum_{\lambda'<\lambda} W^{\lambda'}\alpha$ and it follows by induction that

$$Ff \circ W^\lambda \alpha = Ff \circ \sum_{\lambda'<\lambda} W^{\lambda'}\alpha = \sum_{\lambda'<\lambda} Ff \circ W^{\lambda'}\alpha \leq \sum_{\lambda'<\lambda}(W^{\lambda'}\beta \circ f) =$$
$$\left(\sum_{\lambda'<\lambda} W^{\lambda'}\beta\right) \circ f = \left(\sum_{\lambda'<\lambda} W^{\lambda'}\beta\right) \circ f = W^\lambda\beta \circ f.$$

□

Now for any $\alpha : A \to FA$ define $\alpha^w$ as the least fix point of $W$ greater than or equal to $\alpha$, i.e.

$$\alpha^w := \prod\{\beta \mid \alpha \leq \beta \text{ and } W\beta = \beta\}.$$

Equivalently, we can define $\alpha^w$ as follows (see e.g. [2] for details):

$$\alpha^w := \sum\{W^\lambda\alpha \mid \lambda \in \mathsf{Ord}\}.$$

Clearly, we have $W(\alpha^w) = \alpha^w$ and the following theorem is a direct consequence of Lemma 3.7 and Lemma 3.8.

**Theorem 3.9.** *The assignment $(-)^w$ is a coalgebraic saturator.*

**Definition 3.10.** A relation $R \subseteq A \times B$ between $F$-coalgebras $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ is called a *weak bisimulation* if it is a $(-)^w$-bisimulation.

**Example 3.11.** Given a labelled transition system as a $\mathcal{P}(\Sigma\times\mathcal{I}d)$-coalgebra $\langle A, \alpha\rangle$ the structure $\alpha^w : A \to \mathcal{P}(\Sigma \times A)$ is given by

$$(\sigma, a') \in \alpha^w(a) \iff a \xrightarrow{\tau^*}_\alpha \circ \xrightarrow{\sigma}_\alpha \circ \xrightarrow{\tau^*}_\alpha a' \text{ for } \sigma \neq \tau \text{ and}$$
$$(\tau, a') \in \alpha^w(a) \iff a \xrightarrow{\tau^*}_\alpha a'.$$

Therefore, it coincides with the saturator presented in Example 2.8 and leads to standard definition of weak bisimulation for LTS's.

**Example 3.12.** Given a simple Segala system coalgebra $\langle A, \alpha\rangle$ the saturated structure $\alpha^w$ defined using the operations from Example 3.5 coincides with $\mathfrak{s}\alpha$ from Example 2.10. The reader is referred to Appendix for a proof of this statement.

**Example 3.13.** If we use the operations from Example 3.6 defined in the setting of simple Segala systems then the saturated structure $\alpha^w$ is given by

$$\alpha^w(a) = \{(\sigma, \mu) \mid a \overset{\sigma}{\Longrightarrow} \mu \text{ in } \langle A, \alpha\rangle\},$$

where $a \overset{\sigma}{\Longrightarrow} \mu$ for $\mu \in \mathcal{D}A$ in $\langle A, \alpha\rangle$ means that either $\mu = \delta_a$ and $\sigma = \tau$ or there is a step $(a, \nu)$ in $\langle A, \alpha\rangle$ and a family of measures $\mu_{a'} \in \mathcal{D}A$ for $a' \in A$ such that

$$\mu = \sum_{a'\in A} \nu(a')\mu_{a'}$$

and if $a \xrightarrow{\tau} \nu$ then $a' \overset{\sigma}{\Longrightarrow} \mu_{a'}$ for $a' \in A$ and if $a \xrightarrow{\sigma} \nu$ then $a' \overset{\tau}{\Longrightarrow} \mu_{a'}$ for $a' \in A$. This saturator leads to a definition of weak bisimulation for simple Segala systems instead of probabilistic weak bisimulation. The reader is reffered to [12] for details.

Delay bisimulation for LTS was introduced in [7] and studied in [16]. Here we are going to present an alternative, equivalent definition of this notion for LTS. The proof of equivalence can be found in e.g. [10], Section 4.9.

**Definition 3.14.** Let $\langle A, \Sigma, \to\rangle$ be an LTS. A symmetric relation $R \subseteq A \times A$ is a *delay bisimulation* if the following condition is satisfied. If $(a, b) \in R$ then for $\sigma \neq \tau$ if $a \xrightarrow{\sigma} a'$ then $b \xrightarrow{\tau^*} \circ \xrightarrow{\sigma} b'$ for some $b' \in A$ and $(a', b') \in R$, for $\sigma = \tau$ if $a \xrightarrow{\tau} a'$ then $b \xrightarrow{\tau^*} b'$ for some $b' \in A$ and $(a', b') \in R$.

We will now capture this notion coalgebraically using $\lhd$, $\rhd$ and 1. Consider a family $\{D_A\}_{A \in \mathsf{Set}}$ of mappings defined for any $A \in \mathsf{Set}$ as follows

$$D_A : Hom(A, FA) \to Hom(A, FA); \alpha \mapsto 1 + 1 \lhd \alpha + \alpha \rhd \alpha.$$

As we did for $W$ we may define $D^\lambda$ for $\lambda \in \mathsf{Ord}$. For any $\alpha : A \to FA$ Define $\alpha^d$ as the smallest fixpoint of $D$ greater than or equal to $\alpha$. Equivalently,

$$\alpha^d = \sum_{\lambda \in \mathsf{Ord}} D^\lambda(\alpha).$$

**Example 3.15.** For labelled transition systems and operations $\lhd, \rhd, 1$ from Example 3.4 for any $\alpha : A \to FA$ the structure $\alpha^d$ coincides with the saturator used in Definition 3.14.

**Remark 3.16.** For labelled transition systems the equality $1 \lhd \alpha = \alpha$ implies that $D(\alpha) = 1 + \alpha + \alpha \rhd \alpha$. Therefore, from the point of view of LTS the simplest canditate for $D$ is the operator which assigns to any structure $\alpha : A \to FA$ the structure $1 + \alpha + \alpha \rhd \alpha$. For simple Segala systems such an approach would not be sufficient. Although we could not manage to find a definition of a probabilistic delay bisimulation for simple Segala systems in the literature, the natural way to define delay saturator for simple Segala systems and probabilistic delay bisimulation is the following. For any $a \in A$ and $\sigma \in \Sigma$ we write $a \stackrel{\sigma}{\Longrightarrow}{}^d_P \mu$ whenever $\sigma = \tau$ and $\mu = \delta_a$ or there is a combined step $(a, \nu)$ in $\langle A, \alpha\rangle$ such that if $(\sigma', a') \notin \{\sigma, \tau\} \times A$ then $\nu(\sigma', a') = 0$ and $\mu = \sum_{(\sigma', a') \in \{\sigma, \tau\} \times A} \nu(\sigma', a') \cdot \mu_{(\sigma', a')}$ and if $\sigma' = \tau$ then $a' \stackrel{\sigma}{\Longrightarrow}{}^d_P \mu_{(\sigma', a')}$ otherwise $\sigma' = \sigma$ and $\mu_{(\sigma, a')} = \delta_{a'}$. A brief analysis of the definition of $\Longrightarrow{}^d_P$ leads to a conclusion that the proposed operator assigning to any structure $\alpha$ the structure $1 + \alpha + \alpha \rhd \alpha$ does not suffice. Hence, instead we use $D(\alpha) = 1 + 1 \lhd \alpha + \alpha \rhd \alpha$. In this case the structure $\alpha^d$ coincides with the saturated structure $\mathfrak{s}'\alpha$, where $\mathfrak{s}'\alpha(a) := \{(\sigma, \mu) \mid a \stackrel{\sigma}{\Longrightarrow}{}^d_P \mu\}$. A proof of this statement goes along the lines of the proof of Theorem 6.5 in Appendix and is omitted.

**Theorem 3.17.** *The assignment $(-)^d$ is a coalgebraic saturator.*

**Theorem 3.18.** *For any structure $\alpha$ we have*

$$(\alpha + 1)^w = \alpha^w \text{ and } (\alpha + 1)^d = \alpha^d.$$

*Proof.* We will prove the first equality since the second follows by a similar argument. Since $(-)^w$ is a coalgebraic saturator we have $\alpha^w \leq (1 + \alpha)^w$. Moreover,

$$\begin{aligned}
&W^2(\alpha) = W(1 + \alpha + \alpha \lhd \alpha + \alpha \rhd \alpha) = \\
&1 + (1 + \alpha + \alpha \lhd \alpha + \alpha \rhd \alpha) + (1 + \alpha + \alpha \lhd \alpha + \alpha \rhd \alpha) \lhd (1 + \alpha + \alpha \lhd \alpha + \alpha \rhd \alpha) + \\
&(1 + \alpha + \alpha \lhd \alpha + \alpha \rhd \alpha) \rhd (1 + \alpha + \alpha \lhd \alpha + \alpha \rhd \alpha) \geq \\
&1 + (1 + \alpha) + (1 + \alpha) \lhd (1 + \alpha) + (1 + \alpha) \rhd (1 + \alpha) = W(1 + \alpha).
\end{aligned}$$

Hence, $\alpha^w \geq (1 + \alpha)^w$ which completes the proof. $\square$

**Theorem 3.19.** *For any structure $\alpha : A \to FA$ we have*

$$\begin{aligned}&\alpha^d \leq \alpha^w,\\ &(\alpha \lhd 1)^d = (\alpha \lhd 1)^w,\\ &\alpha^w \lhd 1 = (\alpha \lhd 1)^w,\\ &\alpha^d \lhd 1 = (\alpha \lhd 1)^d.\end{aligned}$$

*Proof.* We will only prove 2nd and 3rd assertion. Note that for a structure $\alpha$ satisfying $\alpha = \alpha \lhd 1$ and $1 \leq \alpha$ we have

$$\begin{aligned}&D(\alpha \lhd 1) = 1 + 1 \lhd (\alpha \lhd 1) + (\alpha \lhd 1) \rhd (\alpha \lhd 1) =\\ &1 + 1 \lhd (\alpha \lhd 1) + (\alpha \lhd 1) \rhd (\alpha \lhd 1) =\\ &1 + \alpha \lhd 1 + 1 \lhd (\alpha \lhd 1) + (\alpha \lhd 1) \rhd (\alpha \lhd 1) \stackrel{(d)}{=}\\ &1 + \alpha \lhd 1 + 1 \rhd (\alpha \lhd 1) + (\alpha \lhd 1) \rhd (\alpha \lhd 1) \stackrel{(h)}{=}\\ &1 + \alpha \lhd 1 + (\alpha \lhd 1) \rhd (\alpha \lhd 1) =\\ &1 + \alpha \lhd 1 + (\alpha \lhd 1) \rhd (\alpha \lhd 1) + (\alpha \lhd 1) \rhd (\alpha \lhd 1) \stackrel{(d)}{=}\\ &1 + \alpha \lhd 1 + (\alpha \lhd 1) \lhd (\alpha \lhd 1) + (\alpha \lhd 1) \rhd (\alpha \lhd 1) = W(\alpha \lhd 1).\end{aligned}$$

Since by the previous lemma $(\alpha + 1)^w = \alpha^w$ and $(\alpha + 1)^d = \alpha^d$ it follows that $(\alpha \lhd 1)^w = (\alpha \lhd 1)^d$.

To prove $\alpha^w \lhd 1 = (\alpha \lhd 1)^w$ note that for any structure $\alpha$ we have

$$\begin{aligned}&W(\alpha) \lhd 1 = (1 + \alpha + \alpha \lhd \alpha + \alpha \rhd \alpha) \lhd 1 =\\ &1 \lhd 1 + \alpha \lhd 1 + (\alpha \lhd \alpha) \lhd 1 + (\alpha \rhd \alpha) \lhd 1 \stackrel{(a)-(c)}{=}\\ &1 + \alpha \lhd 1 + \alpha \lhd (\alpha \lhd 1) + (\alpha \lhd 1) \rhd \alpha \stackrel{(a)}{=}\\ &1 + \alpha \lhd 1 + (\alpha \lhd 1) \lhd (\alpha \lhd 1) + (\alpha \lhd 1) \rhd (\alpha \lhd 1) = W(\alpha \lhd 1).\end{aligned}$$

Now by (d) it follows that

$$(\alpha \lhd 1)^w = \sum_{\lambda \in \mathsf{Ord}} W^\lambda(\alpha \lhd 1) = \sum_{\lambda \in \mathsf{Ord}} W^\lambda(\alpha) \lhd 1 = \left(\sum_{\lambda \in \mathsf{Ord}} W^\lambda(\alpha)\right) \lhd 1 = \alpha^w \lhd 1.$$

□

For any $F$-coalgebra $\langle A, \alpha \rangle$ let $\approx_d$ and $\approx_w$ denote the relations $\approx_{(-)^d}$ and $\approx_{(-)^w}$ respectively.

**Theorem 3.20.** *For an $F$-coalgebra $\langle A, \alpha \rangle$ and $a, b \in A$ we have*

$$a \approx_d b \implies a \approx_w b.$$

**Remark 3.21.** Our setting also allows us to define other equivalences. For any structure $\alpha : A \to FA$ consider the structure $P(\alpha) = 1 \lhd \alpha$. We see that $\alpha \leq P(\alpha)$. Take $\alpha^p = \sum_{\lambda \in \mathsf{Ord}} P^\lambda(\alpha)$. The assignment $(-)^p$ is a coalgebraic saturator. For LTS, $\alpha^p = \alpha$, so $(-)^p$-bisimulation is a strong bisimulation. For simple Segala systems and operations from Example 3.5 for a structure $\alpha : A \to \mathcal{P}(\Sigma \times \mathcal{D}A)$ and $a \in A$ we have $(\sigma, \mu) \in \alpha^p(a)$ if and only if there is a family of positive numbers $\{p_i\}_{i \in I}$ such that $i \in I$ and $\mu = \sum_{i \in I} p_i \mu_i$ for $(\sigma, \mu_i) \in \alpha(a)$ for any $i \in I$. Hence,

the notion of $(-)^p$-bisimulation coincides with the notion of strong probabilistic bisimulation (see [12] for details).

## 4. Algebraic structure on final coalgebra

In this section we assume that a $\mathsf{Set}$-endofunctor $F$ admits the terminal $F$-coalgebra $\langle Z, \zeta\rangle$. In [1] we introduced a category $\mathsf{Set}^w_F$ whose objects are $F$-coalgebras and whose morphisms are mappings $f : A \to B$ between $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ for which the relation $graph(f) = \{(a, f(a)) \mid a \in A\}$ is a weak bisimulation. We formulated some results concerning a weak coinduction principle. Let $\langle Z^w, \zeta_{|Z^w}\rangle$ be the greatest subcoalgebra of the terminal coalgebra $\langle Z, \zeta\rangle$ such that $(\zeta_{|Z^w})^w = \zeta_{|Z^w}$. This subcoalgebra exists by [1].

**Theorem 4.1.** [1] *If $F$ considered as a $\mathsf{Set}$-endofunctor weakly preserves pullbacks then the coalgebra $\langle Z^w, \zeta_{|Z^w}\rangle$ is a terminal object in $\mathsf{Set}^w_F$.*

**Theorem 4.2.** [1] *Let $F$ considered as a $\mathsf{Set}$-endofunctor weakly preserve pullbacks. For any $F$-coalgebra $\langle A, \alpha\rangle$ and any elements $a, b \in A$ we have*

$$a \approx_w b \iff [\![a]\!]^w_\alpha = [\![b]\!]^w_\alpha,$$

*where $[\![-]\!]^w_\alpha = [\![-]\!]_{\zeta^w} \circ [\![-]\!]_\alpha$ and $[\![-]\!]_\beta$ denotes the unique homomorphism from a coalgebra $\langle B, \beta\rangle$ to $\langle Z, \zeta\rangle$ in $\mathsf{Set}_F$.*

The aim of this section is to characterize elements from $Z$ which belong to $Z^w$ (or $Z^d$) in terms of $\triangleleft$, $\triangleright$ and $1$.

Recall that the structure $\zeta : Z \to FZ$ is a bijection. For any $z, z' \in Z$ we define:

- $z \leq z'$ whenever $\zeta(z) \leq \zeta(z')$ in $FZ$,
- $\tau z := \zeta^{-1}(1_Z(z))$,
- $z \triangleleft z := \zeta^{-1}([\zeta \triangleleft \zeta](z))$,
- $z \triangleright z := \zeta^{-1}([\zeta \triangleright \zeta](z))$,
- $1 \triangleleft z := \zeta^{-1}([1 \triangleleft \zeta](z))$.

**Theorem 4.3.** *Any element $z \in Z^w$ satisfies the following equation:*

$$z = \tau z + z + z \triangleleft z + z \triangleright z.$$

*Proof.* Since $\zeta^w = 1 + \zeta^w + \zeta^w \triangleleft \zeta^w + \zeta^w \triangleright \zeta^w$ and $\zeta^w_{|Z^w} = \zeta_{|Z^w}$ we have

$$\zeta_{|Z^w} = 1_{|Z^w} + \zeta_{|Z^w} + \zeta_{|Z^w} \triangleleft \zeta_{|Z^w} + \zeta_{|Z^w} \triangleright \zeta_{|Z^w}$$

This means that for any $z \in Z^w$ the following equality is true:

$$\zeta(z) = 1(z) + \zeta(z) + (\zeta \triangleleft \zeta)(z) + (\zeta \triangleright \zeta)(z).$$

Hence,

$$z = \tau z + z + z \triangleleft z + z \triangleright z.$$

□

**Theorem 4.4.** *Let $\langle Z', \zeta_{|Z'}\rangle$ be a subcoalgebra of $\langle Z, \zeta\rangle$ such that for any $z \in Z'$ we have*

$$z = \tau z + z + z \triangleleft z + z \triangleright z.$$

*Then $\langle Z', \zeta_{|Z'}\rangle$ is a subcoalgebra of $\langle Z^w, \zeta_{|Z^w}\rangle$.*

*Proof.* Since $\langle Z^w, \zeta_{|Z^w}\rangle$ is the greatest subcoalgebra of $\langle Z, \zeta\rangle$ whose structure is saturated (see [1] for details) it is enough to show that $\zeta^w_{|Z'} = \zeta_{|Z'}$. By the assumed equality satisfied by any $z \in Z'$ it follows that for $z \in Z'$ we have

$$\zeta(z) = 1(z) + \zeta(z) + (\zeta \lhd \zeta)(z) + (\zeta \rhd \zeta)(z).$$

This precisely means that $\zeta_{|Z'} = 1_{|Z'} + \zeta_{|Z'} + \zeta_{|Z'} \lhd \zeta_{|Z'} + \zeta_{|Z'} \rhd \zeta_{|Z'}$. Hence, $\zeta^w_{|Z'} = \zeta_{|Z'}$. □

All results from [1] about weak coinduction principle are true for any saturator, in particular for $(-)^d$. Consider the greatest subcoalgebra $\langle Z^d, \zeta_{|Z^d}\rangle$ of $\langle Z, \zeta\rangle$ such that

$$(\zeta_{|Z^d})^d = \zeta_{|Z^d}.$$

**Theorem 4.5.** *Any element $z \in Z^d$ satisfies the following equation:*

$$z = \tau z + 1 \lhd z + z \rhd z.$$

## 5. Saturator approximants

Recall that the definition of $(-)^w$ and $(-)^d$ requires from the order on $FX$ to be join complete for any set $X$. In some cases this is a too strong an assumption. For instance if we consider locally finite labelled transistion systems, i.e. coalgebras of the type $F = \mathcal{P}_{fin}(\Sigma \times \mathcal{I}d)$, then it is impossible to put them into our setting because the underlying functor is not endowed with a complete partial order and it is impossible to define the saturated structures

$$\alpha^w : A \to \mathcal{P}_{fin}(\Sigma \times A) \text{ or } \alpha^d : A \to \mathcal{P}_{fin}(\Sigma \times A)$$

for any $\alpha : A \to \mathcal{P}_{fin}(\Sigma \times A)$.

For sake of this section let us assume that our $F$ has a subfunctor $F' : \mathsf{Set} \to \mathsf{JSLat}$ whose codomain $\mathsf{JSLat}$ is the category of join semilattices and homomorphisms between them and not neccessarily complete join semilattices. Moreover we assume for any set $A$ the set of structures $Hom(A, F'A)$ is closed under the binary operations

$$\lhd, \rhd : Hom(A, FA)^2 \to Hom(A, FA)$$

introduced in Section 3 and $1_A \in Hom(A, F'A) \subseteq Hom(A, FA)$. In other words, for any $\alpha : A \to F'A$ the images $W(\alpha) : A \to F'A$ and $D(\alpha) : A \to F'A$.

**Example 5.1.** Consider $F = \mathcal{P}(\Sigma \times \mathcal{I}d)$, $F' = \mathcal{P}_{fin}(\Sigma \times \mathcal{I}d)$ and $\lhd, \rhd$, 1 defined as in Example 3.4. The operations satisfy the above requirement. For $F = \mathcal{P}(\Sigma \times \mathcal{D})$, $F' = \mathcal{P}_{fin}(\Sigma \times \mathcal{D}_{fin})$, where $\mathcal{D}_{fin}$ denotes a subfunctor of $\mathcal{D}$ defined for any set $X$ by

$$\mathcal{D}_{fin}X = \{f : X \to [0,1] \mid f \in \mathcal{D}X \text{ and } supp(f) < \omega\},$$

the operations $\lhd$, $\rhd$, 1 taken from Example 3.6 also satisfy the above requirement. Now if $F = \mathcal{P}(\Sigma \times \mathcal{D})$, $F' = \mathcal{P}_{fin}(\Sigma \times \mathcal{D}_{fin})$ and $\lhd, \rhd$ with 1 taken from Example 3.5 then we see that the set $Hom(A, \mathcal{P}_{fin}(\Sigma \times \mathcal{D}_{fin}A))$ is not closed under $\lhd$ or $\rhd$.

Our aim is to introduce a definition of a weak (delay) bisimulation for $F'$-coalgebras without the need to speak about $F$-coalgebraic structures and join-complete partial order.

For any family of coalgebraic structures $\{\alpha_i : A \to F'A\}_{i\in I}$ define

$$\ll \{\alpha_i\}_{i\in I} \gg := \{\beta : A \to F'A \mid \forall a \in A \quad \exists i \in I \text{ such that } \beta(a) = \alpha_i(a)\}.$$

The key idea behind defining weak (or delay) bisimulation for $F'$-coalgebras is to replace the saturated structure $\alpha^w$ (or $\alpha^d$) with a structure from the set

$$\ll \{W^n(\alpha) \mid n \in \mathbb{N}\} \gg \text{ (resp. } \ll \{D^n(\alpha) \mid n \in \mathbb{N}\} \gg).$$

To be more precise we present the following definition.

**Definition 5.2.** A relation $R \subseteq A \times B$ is called an *approximated weak bisimulation* between $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ if there are structures $\gamma_1, \gamma_2 : R \to FR$ satisfying:

- $\alpha \circ \pi_1 = F(\pi_1) \circ \gamma_1$ and $\beta' \circ \pi_2 \geq F(\pi_2) \circ \gamma_1$, for some $\beta' \in \ll \{W^n(\beta) \mid n \in \mathbb{N}\} \gg$,
- $\beta \circ \pi_2 = F(\pi_2) \circ \gamma_2$ and $\alpha' \circ \pi_1 \geq F(\pi_1) \circ \gamma_2$, for some $\alpha' \in \ll \{W^n(\alpha) \mid n \in \mathbb{N}\} \gg$.

We define *approximated delay bisimulatio*n in a similar manner by replacing $W$ with $D$. We will only state theorems about weak and approximated weak bisimulation. They will remain true for approximated delay bisimulation.

**Theorem 5.3.** *If $R$ is an approximated weak bisimulation between two $F'$-coalgebras $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ then $R$ is a weak bisimulation between these structures.*

**Theorem 5.4.** *Assume that for any set $X$ the poset $F'X$ consists only of compact elements, i.e. if $\vec{a} \leq \sum_{i \in I} \vec{b}_i$ for $\vec{a}, \vec{b}_i \in F'X$ then there is a finite subset $I_0 \subseteq I$ such that $\vec{a} \leq \sum_{i \in I_0} \vec{b}_i$. Moreover, assume that for any $f : X \to Y$ and $\vec{x} \in FX$ if $Ff(\vec{x}) = \vec{y} \in F'Y$ then there is $\vec{x}' \leq \vec{x}$ such that $\vec{x}' \in F'X$ with $Ff(\vec{x}') = \vec{y}$. Moreover assume that for any $\alpha : A \to FA$ the saturated structure $\alpha^w = \sum_{n \in \mathbb{N}} W^n(\alpha)$. In this case any weak bisimulation between two $F'$-coalgebras is an approximated weak bisimulation.*

*Proof.* Let $R \subseteq A \times B$ be a weak bisimulation between two $F'$-coalgebras $\langle A, \alpha\rangle$ and $\langle B, \beta\rangle$ considered as $F$-coalgebras. In particular this means that there is $\gamma_1 : R \to FR$ satisfying:

$$\alpha \circ \pi_1 = F(\pi_1) \circ \gamma_1 \text{ and } F(\pi_2) \circ \gamma_1 \leq \beta^w \circ \pi_2.$$

For $(a, b) \in R$ put $\vec{r}_{(a,b)} := \gamma_1(a, b)$. Note that for $\pi_1 : R \to A$ we have $F(\pi_1)(\vec{r}_{(a,b)}) = \alpha(a)$ and $\alpha(a) \in F'A$. By our assumption it follows that there is $\vec{r}'_{(a,b)} \in F'R$ such that $F(\pi_1)(\vec{r}'_{(a,b)}) = F'(\pi_1)(\vec{r}'_{(a,b)}) = \alpha(a)$ and $\vec{r}'_{(a,b)} \leq \vec{r}_{(a,b)}$. Now, for any $(a, b) \in R$ define

$$\gamma_1' : R \to F'R; (a, b) \mapsto \vec{r}'_{(a,b)}$$

and note that

- $\alpha \circ \pi_1(a, b) = \alpha(a) = F'(\pi_1)(\vec{r}'_{(a,b)}) = F'(\pi_1) \circ \gamma_1'(a, b)$ and
- $F(\pi_2)(\vec{r}'_{(a,b)}) \leq F(\pi_1)(\vec{r}_{(a,b)}) \leq \beta^w(b)$.

Since $\beta^w = \sum_{n \in \mathbb{N}} W^n(\beta)$ it follows that $F(\pi_2)(\vec{r}'_{(a,b)}) \leq \sum_{n \in \mathbb{N}} W^n(\beta)(b)$.

By compactness of elements from $F'X$ and the inclusions $F(\pi_2)(\vec{r}'_{(a,b)}) \in F'B$ and $W^n(\beta)(b) \in F'B$ for any $n \in \mathbb{N}$ it follows that there is a finite set of natural numbers $\{n_1, \ldots, n_k\}$ such that

$$F(\pi_2)(\vec{r}'_{(a,b)}) \leq W^{n_1}(\beta)(b) \vee \ldots \vee W^{n_k}(\beta)(b).$$

Let $m_b$ be the greatest integer among $n_1, \ldots, n_k$. Then $F(\pi_2)(\vec{r}'_{(a,b)}) \leq W^{m_b}(\beta)(b)$. If we define $\beta' : B \to F'B$ by putting $\beta'(b) = W^{m_b}(\beta)(b)$ then

- $\alpha \circ \pi_1(a, b) = \alpha(a) = F'(\pi_1)(\vec{r}'_{(a,b)}) = F'(\pi_1) \circ \gamma_1'(a, b)$ and

- $F'(\pi_2) \circ \gamma_1' \leq \beta' \circ \pi_2$ and $\beta' \in\ll \{W^n(\beta) \mid n \in \mathbb{N}\} \gg$.

Similarily we prove existence of $\gamma_2'$ and $\alpha'$ satisfying the desired properties. □

**Example 5.5.** For locally finite labelled transition systems, i.e. coalgebras over $F = \mathcal{P}_{fin}(\Sigma \times \mathcal{I}d)$ the notion of an approximated weak bisimulation is equivalent to the notion of weak bisimulation between locally finite LTS's considered as $\mathcal{P}(\Sigma \times \mathcal{I}d)$ coalgebras.

## 6. Future work

Other types of equivalences between different transistion systems are also very important from the point of view of applications, e.g. branching bisimulation [3]. It would be very interesting to see if the algebraic structure introduced in this paper allows to define branching bisimulation for the considered coalgebras. If so, what are its properties with respect to e.g. weak and delay bisimulation defined here.

In Section 4 we defined an algebraic structure on the carrier of the final coalgebra and related it to weak and delay bisimulation. The results presented there should be considered the starting point of research towards equational characterizations of different equivalences (e.g. weak and delay bisimilarities) for coalgebras.

Finally, a purely algebraic structure emerges from our results. Namely, an algebra $(A, +, \triangleleft, \triangleright, (-)^d, (-)^w, 0, 1)$, where $(A, +)$ is a join semilattice, whose smallest element is 0, $\triangleleft, \triangleright$ are binary operations, $(-)^d$ and $(-)^w$ unary operations, and 1 a constant satisfying properties (a)-(e) from Section 3 (here second equality from (d) is required to hold only for finite joins) and additionally satisfy:

$$
\begin{aligned}
& a \leq a' \text{ and } b \leq b' \implies a \diamond b \leq a' \diamond b' \text{ for } \diamond \in \{\triangleleft, \triangleright\}, \\
& a \leq a^w \\
& a^w = 1 + a^w + a^w \triangleleft a^w + a^w \triangleright a^w, \\
& a \leq b \text{ and } b = 1 + b + b \triangleleft b + b \triangleright b \implies a^w \leq b, \\
& a \leq a^d \\
& a^d = 1 + 1 \triangleleft a^d + a^d \triangleright a^d, \\
& a \leq b \text{ and } b = 1 + 1 \triangleleft b + b \triangleright b \implies a^d \leq b.
\end{aligned}
$$

Such algebras bare some resemblance to Kleene algebras [5]. It would be interesting to examine their properties.

*Acknowledgements.* I would like to thank Jan Rutten for many interesting comments and hospitality in CWI, Amsterdam, where part of the work on this paper was completed. I am very grateful to F. Bonchi, M. Bonsangue, H. Hansen, B. Klin and A. Silva for various suggestions for improvement of my results and for future work.

## References

[1] Brengos, T.: Weak bisimulations for coalgebras over ordered functors. Proc. IFIP TCS 2012, LNCS Vol. 7604 (2012)

[2] Davey, B. A., Priestley, H.A.: Introduction to lattices and order. Cambridge University Press (200)

[3] van Glabbeek, R. J., Weijland, W. P.: Branching time and abstraction in bisimulation semantics. J. ACM, 43(3):555600, 1996.

[4] Gumm, H.P.: Elements of the general theory of coalgebras. LUATCS 99, Rand Afrikaans University, Johannesburg (1999)
[5] Kozen, D.: On Kleene algebras and closed semirings. In: Lecture Notes in Computer Science vol. 452 pp. 26-47
[6] Milner, R.: A Calculus of Communicating Systems. LNCS 92 (1980)
[7] Milner, R.: A modal characterisation of observable machine-behaviour. Proceedings CAAP 81, LNCS 112, Springer-Verlag, pp. 25-34. (1981)
[8] Rothe, J.: A syntactical approach to weak (bi)-simulation for coalgebras. In: L. Moss (ed.) Proc. CMCS02. ENTCS, vol 65, pp. 270-285 (2002)
[9] Rothe, J., Masulović D.: Towards weak bisimulation for coalgebras. In: Proc. Categorical Methods for Concurrency, Interaction and Mobility, ENTCS, vol. 68 issue 1, pp. 32-46 (2002)
[10] Sangiorgi, D.: Introduction to Bisimulation and Coinduction. Cambridge University Press (2011)
[11] Segala, R., Lynch, N.: Probabilistic simulations for probabilistic processes. In: B. Jonsson, J. Parrow (eds.) Proc. CONCUR94, pp. 481 496. LNCS 836 (1994)
[12] Segala, R.: Modeling and verification of randomized distributed real-time systems. Ph.D. thesis, MIT (1995)
[13] Sokolova, A., de Vink, E., Woracek, H: Coalgebraic Weak Bisimulation for Action-Type Systems. Sci. Ann. Comp. Sci. 19: 93-144 (2009)
[14] Rutten, J. J. M. M.: A note on coinduction and weak bisimilarity for while programs. Theoretical Informatics and Applications (RAIRO) 33, 393400 (1999)
[15] Rutten, J. J. M. M.: Universal coalgebra: a theory of systems. Theor. Comput. Sci. 249(1): 3-80 (2000)
[16] Weijland, W. P.: Synchrony and Asynchrony in Process Algebra. Ph.D. thesis, University of Amsterdam, 1989.

## Appendix

**Theorem 6.1.** *The operations $\lhd, \rhd$, $0$ and $1$ from Example 3.5 satisfy properties (a)-(h) from Section 3.*

*Proof.* Note that for any $\alpha : A \to \mathcal{P}(\Sigma \times \mathcal{D}A)$ the structure $\alpha \lhd 1 : A \to \mathcal{P}(\Sigma \times \mathcal{D}A)$ is given by:

$$(\sigma, \mu) \in \alpha \lhd 1(a) \iff \sigma = \tau \text{ and } (\tau, \mu) \in \alpha(a).$$

Hence, property (a) holds. To see (b) is true take a structure $\alpha' : A \to \mathcal{P}(\Sigma \times \mathcal{D}A)$ and note that for any $a \in A$ we have

$$(\sigma, \mu) \in (\alpha \lhd 1) \rhd \alpha'(a) \iff \sigma = \tau \text{ and } \mu = \sum_{a' \in A} \nu(\tau, a') \cdot \mu'_a,$$

for a combined step $(a, \nu)$ in $\alpha'$ with only $\tau$ transitions and $a' \xrightarrow{\tau}_{\alpha} \mu'_a$ for any $a' \in A$. Hence, $(\sigma, \mu) \in (\alpha \lhd 1) \rhd \alpha'(a) \iff (\sigma, \mu) \in (\alpha \rhd \alpha') \lhd 1(a)$. To prove the second identity from $(b)$ note that $(\sigma, \mu) \in (\alpha \lhd \alpha') \lhd 1(a)$ if and only if $\sigma = \tau$ and $(\sigma, \mu) \in \alpha \lhd \alpha'(a)$. This is true if and only if $(\sigma, \mu) \in \alpha \lhd (\alpha' \lhd 1)(a)$. Properties in (c) are obvious. To prove first equality from (d) it is enough to note that for any structures $\beta, \beta' : A \to \mathcal{P}(\Sigma \times \mathcal{D}A)$ with only $\tau$ transitions we have:

$$\beta \lhd \beta' = \beta \rhd \beta'.$$

The second equality from (d) holds since for any family of structures $\alpha_i : A \to \mathcal{P}(\Sigma \times \mathcal{D}A)$ the structure $\sum_{i \in I} (\alpha_i \lhd 1)$ contains only $\tau$ transitions from the union of $\alpha_i$. Hence,

$$\sum_{i \in I} (\alpha_i \lhd 1) = (\sum_{i \in I} \alpha_i) \lhd 1.$$

Properties $(e)$ and (f) are obvious. To prove (g) consider coalgebraic homomorphism $f : A \to B$ between structures $\alpha : A \to \mathcal{P}(\Sigma \times \mathcal{D}A)$, $\beta : B \to \mathcal{P}(\Sigma \times \mathcal{D}B)$ and $\alpha' : A \to \mathcal{P}(\Sigma \times \mathcal{D}A)$, $\beta' : B \to \mathcal{P}(\Sigma \times \mathcal{D}B)$. For $a \in A$ consider a pair $(\sigma, \mu) \in \beta \lhd \beta'(f(a))$. This means that $\mu = \sum_{b' \in B} \nu(\sigma, b') \cdot \mu_{b'}$ for a combined step $(f(a), \nu)$ in $\langle B, \beta' \rangle$ such that $\nu(\sigma', b') = 0$ for any $b' \in B$ and $\sigma' \neq \sigma$ and $\mu_{b'} \in \mathcal{D}B$ with $b' \xrightarrow{\tau}_{\beta} \mu_{b'}$. Therefore there is a family of positive numbers $\{p_i\}_{i \in I}$ with $\sum_{i \in I} p_i = 1$ and a family of measures $\{\mu_i\}_{i \in I} \subseteq \mathcal{D}B$ satisfying

$$\nu = \sum_{i \in I} p_i \cdot \mu_i \text{ and } f(a) \xrightarrow{\sigma}_{\beta'} \mu_i.$$

Since $f : A \to B$ is a homomorphism between $\langle A, \alpha \rangle$, $\langle B, \beta \rangle$ and $\langle A, \alpha' \rangle$, $\langle B, \beta' \rangle$ for any $a \in A$ we have

$$f(a) \xrightarrow{\sigma}_{\beta'} \mu_i \iff \mu_i(b) = \sum_{a \in f^{-1}(\{b\})} \mu'_i(a) \text{ for } \mu'_i \in \mathcal{D}A \text{ s.t. } a \xrightarrow{\sigma}_{\alpha'} \mu'_i,$$

$$b' \xrightarrow{\tau}_{\beta} \mu_{b'} \iff \mu_{b'}(b) = \sum_{a \in f^{-1}(\{b\})} \mu_{a'}(a) \text{ for } \mu'_{a'} \in \mathcal{D}A \text{ s.t. } a' \xrightarrow{\tau}_{\alpha'} \mu_{a'}$$

for $a' \in f^{-1}(\{b'\})$.

Consider the combined step $(a, \nu')$ in $\langle A, \alpha' \rangle$, where $\nu' = \sum_{i \in I} p_i \cdot \mu'_i$ and put

$$\mu' = \sum_{a' \in A} \nu'(\sigma, a') \cdot \mu_{a'}$$

Then $a \xrightarrow{\sigma}_{\alpha \lhd \alpha'} \mu'$ and for any $b'' \in B$ we have

$$\sum_{a'' \in f^{-1}(\{b''\})} \mu'(a'') = \sum_{a'' \in f^{-1}(\{b'\})} \left( \sum_{a' \in A} \nu'(\sigma, a') \cdot \mu_{a'} \right)(a'') =$$
$$\sum_{a'' \in f^{-1}(\{b''\})} \sum_{a' \in A} \nu'(\sigma, a') \cdot \mu_{a'}(a'') = \sum_{a' \in A} \nu'(\sigma, a') \cdot \sum_{a'' \in f^{-1}(\{b''\})} \mu_{a'}(a'') =$$
$$\sum_{b' \in B} \sum_{a' \in f^{-1}(\{b'\})} \nu'(\sigma, a') \cdot \sum_{a'' \in f^{-1}(\{b''\})} \mu_{a'}(a'') =$$
$$\sum_{b' \in B} \sum_{a' \in f^{-1}(\{b'\})} \sum_{i \in I} p_i \cdot \mu_i'(a') \cdot \sum_{a'' \in f^{-1}(\{b''\})} \mu_{a'}(a'') =$$
$$\sum_{b' \in B} \sum_{a' \in f^{-1}(\{b'\})} \sum_{i \in I} p_i \cdot \mu_i'(a') \cdot \mu_{b'}(b'') =$$
$$\sum_{b' \in B} \sum_{i \in I} p_i \cdot \left( \sum_{a' \in f^{-1}(\{b'\})} \mu_i'(a') \right) \cdot \mu_{b'}(b'') = \sum_{b' \in B} \sum_{i \in I} p_i \cdot \mu_i(b') \cdot \mu_{b'}(b'') =$$
$$= \sum_{b' \in B} \nu(\sigma, b') \cdot \mu_{b'}(b'') = \mu(b'').$$

Hence, $f : A \to B$ is a homomorphism between $\langle A, \alpha \lhd \alpha' \rangle$ and $\langle B, \beta \lhd \beta' \rangle$. In a similar way we prove the statement for the operation $\rhd$. Finally, rule (h) is proved using the similar reasoning as in (g). $\square$

**Lemma 6.2.** *For simple Segala system coalgebra $\langle A, \alpha \rangle$ the saturated structure $\mathfrak{s}\alpha$ defined in Example 2.10 satisfies:*

$$\alpha \leq \mathfrak{s}\alpha,$$
$$\mathfrak{s}\alpha = 1 + \mathfrak{s}\alpha + \mathfrak{s}\alpha \lhd \mathfrak{s}\alpha + \mathfrak{s}\alpha \rhd \mathfrak{s}\alpha.$$

*Proof.* It is obvious that $\alpha \leq \mathfrak{s}\alpha$. To see the second property is true it is enough to show $1 \leq \mathfrak{s}\alpha$, $\mathfrak{s}\alpha \lhd \mathfrak{s}\alpha \leq \mathfrak{s}\alpha$ and $\mathfrak{s}\alpha \rhd \mathfrak{s}\alpha \leq \mathfrak{s}\alpha$. By definition of $\mathfrak{s}\alpha$ it follows directly that $1 \leq \mathfrak{s}\alpha$. To see that $\mathfrak{s}\alpha \lhd \mathfrak{s}\alpha \leq \mathfrak{s}\alpha$ for any $a \in A$ take $(\sigma, \mu) \in \mathfrak{s}\alpha \lhd \mathfrak{s}\alpha(a)$. Hence, there is a combined step $(a, \nu)$ in $\mathfrak{s}\alpha$ such that $\nu(\sigma', a') = 0$ for any $\sigma' \neq \sigma$ and $a' \in A$ and

$$\mu = \sum_{a' \in A} \nu(\sigma, a') \cdot \mu_{a'} \text{ for } (\tau, \mu_{a'}) \in \mathfrak{s}\alpha(a').$$

This means that there is a family of positive numbers $\{p_i\}_{i \in I}$ such that $\sum_{i \in I} p_i = 1$ and a family of measures $\{\mu_i\}_{i \in I} \subseteq \mathcal{D}A$ with $(\sigma, \mu_i) \in \mathfrak{s}\alpha(a)$ for any $i \in I$ and

$$\nu(\sigma, a') = \sum_{i \in I} p_i \cdot \mu_i(a').$$

The inclusion $(\sigma, \mu_i) \in \mathfrak{s}\alpha(a)$ implies that for any $i \in I$ there is a combined step $(a, \nu_i)$ in $\alpha$ such that

$$\mu_i = \sum_{a'' \in A, \sigma' \in \{\sigma, \tau\}} \nu_i(\sigma', a'') \mu_{i, \sigma', a''} \text{ and}$$

$\nu_i(\sigma', a'') = 0$ for $\sigma' \notin \{\sigma, \tau\}$, $a'' \in A$. Therefore, we have

$$\mu = \sum_{a' \in A} \nu(\sigma, a') \cdot \mu_{a'} = \sum_{a' \in A} \left( \sum_{i \in I} p_i \mu_i(a') \right) \cdot \mu_{a'} = \sum_{i \in I} p_i \sum_{a' \in A} \mu_i(a') \cdot \mu_{a'} =$$
$$\sum_{i \in I} p_i \cdot \sum_{a' \in A} \left( \sum_{a'' \in A, \sigma' \in \{\sigma, \tau\}} \nu_i(\sigma', a'') \mu_{i,\sigma',a''}(a') \right) \cdot \mu_{a'} =$$
$$\sum_{i \in I} p_i \cdot \sum_{a'' \in A, \sigma' \in \{\sigma, \tau\}} \nu_i(\sigma', a'') \cdot \sum_{a' \in A} \mu_{i,\sigma',a''}(a') \cdot \mu_{a'} =$$
$$\sum_{i \in I} p_i \cdot \sum_{a'' \in A, \sigma' \in \{\sigma, \tau\}} \nu_i(\sigma', a'') \mu'_{i,\sigma',a''},$$

where $\mu'_{i,\sigma',a''} = \sum_{a' \in A} \mu_{i,\sigma',a''}(a') \cdot \mu_{a'}$ and $(\kappa, \mu'_{i,\sigma',a''}) \in \mathfrak{s}\alpha(a'')$ for

$$\kappa := \textbf{ if } \sigma' = \tau \textbf{ then } \sigma \textbf{ otherwise } \tau.$$

Hence, $\mu = \sum_{i \in I} p_i \xi_i$ for $(\sigma, \xi_i) \in \mathfrak{s}\alpha(a)$ and therefore $\mu \in \mathfrak{s}\alpha(a)$. Similarily we prove $\mathfrak{s}\alpha \triangleright \mathfrak{s}\alpha \leq \mathfrak{s}\alpha$. $\square$

**Lemma 6.3.** *For simple Segala system coalgebra $\langle A, \alpha \rangle$ and the saturated structures $\mathfrak{s}\alpha$ and $\alpha^w$ from Example 2.10 and Example 3.12 we have*

$$\alpha^w \leq \mathfrak{s}\alpha.$$

*Proof.* By definition, the structure $\alpha^w$ is the smallest structure such that $\alpha \leq \alpha^w$ and $\alpha^w = 1 + \alpha^w + \alpha^w \triangleleft \alpha^w + \alpha^w \triangleright \alpha^w$. By the previous lemma the structure $\mathfrak{s}\alpha$ satisfies the same properties. Hence, $\alpha^w \leq \mathfrak{s}\alpha$. $\square$

**Lemma 6.4.** *For simple Segala system coalgebra $\langle A, \alpha \rangle$ and the saturated structures $\mathfrak{s}\alpha$ and $\alpha^w$ from Example 2.10 and Example 3.12 we have*

$$\alpha^w \geq \mathfrak{s}\alpha.$$

*Proof.* It is enough to show that for any $a \in A$, $\sigma \in \Sigma$ and

$$M_\sigma(a) = \{\mu \in \mathcal{D}A \mid (\sigma, \mu) \in \alpha^w\}$$

the family $\{M_\sigma(a)\}_{\sigma,a}$ satisfies the properties listed in Example 2.10. By definition of $\alpha^w$ the set $M_\tau(a)$ contains the measure $\delta_a$ for any $a \in A$. Moreover, let a measure $\mu \in \mathcal{D}A$ be given by

$$\mu = \sum_{a' \in A, \sigma' \in \{\sigma, \tau\}} \nu(\sigma', a') \mu_{(\sigma', a')},$$

where $(a, \nu)$ is a combined step in $\langle A, \alpha \rangle$ such that $\nu(\sigma', a') = 0$ for $\sigma' \notin \{\sigma, \tau\}$, $a' \in A$ and $\mu_{(\sigma', a')} \in M_\kappa(a')$ for $\kappa :=$ **if** $\sigma' = \tau$ **then** $\sigma$ **otherwise** $\tau$. Since $(a, \nu)$ is a combined step in $\langle A, \alpha \rangle$ there is a family $\{p_i\}_{i \in I}$ of positive numbers such that $\sum_{i \in I} p_i = 1$ and a family $\{(a, \mu_i)\}$ of steps in $\langle A, \alpha \rangle$ such that $\nu = \sum_{i \in I} p_i \cdot \mu_i$. We

have the following:

$$\mu = \sum_{a'\in A,\sigma'\in\{\sigma,\tau\}} \nu(\sigma',a')\mu_{(\sigma',a')} = \sum_{\sigma'\in\{\sigma,\tau\}}\sum_{a'\in A} \nu(\sigma',a')\mu_{(\sigma',a')} =$$
$$\sum_{\sigma'\in\{\sigma,\tau\}}\sum_{a'\in A}\sum_{i\in I} p_i\cdot\mu_i(\sigma',a')\mu_{(\sigma',a')} =$$
$$\sum_{a'\in A}\sum_{i\in I} p_i\cdot\mu_i(\sigma,a')\mu_{(\sigma,a')} + \sum_{a'\in A}\sum_{i\in I} p_i\cdot\mu_i(\tau,a')\mu_{(\tau,a')} =$$
$$\sum_{a'\in A}\sum_{i\in I_1} p_i\cdot\mu_i(a')\mu_{(\sigma,a')} + \sum_{a'\in A}\sum_{i\in I_2} p_i\cdot\mu_i(a')\mu_{(\tau,a')} =$$
$$q_1\cdot\sum_{a'\in A}\nu_1(\sigma,a')\mu_{(\sigma,a')} + q_2\cdot\sum_{a'\in A}\nu_2(\tau,a')\mu_{(\tau,a')},$$

where

$$I_1 = \{i\in I \mid \mu_i(\sigma,a')\neq 0 \text{ for some } a'\in A\},$$
$$I_2 = \{i\in I \mid \mu_i(\tau,a')\neq 0 \text{ for some } a'\in A\},$$
$$q_1 := \textbf{ if } I_1\neq\varnothing \textbf{ then } \frac{1}{\sum_{i\in I_1} p_i} \textbf{ else } 0$$
$$q_2 := \textbf{ if } I_2\neq\varnothing \textbf{ then } \frac{1}{\sum_{i\in I_2} p_i} \textbf{ else } 0$$
$$\nu_1 := \sum_{i\in I_1}\frac{p_i}{q_1}\mu_i,$$
$$\nu_2 := \sum_{i\in I_2}\frac{p_i}{q_2}\mu_i.$$

We see that $\nu_1$ and $\nu_2$ are combined steps in $\langle A,\alpha\rangle$ and both of the measures

$$\sum_{a'\in A}\nu_1(\sigma,a')\mu_{(\sigma,a')}, \qquad \sum_{a'\in A}\nu_2(\tau,a')\mu_{(\tau,a')}$$

belong to $M_\sigma(a)$. Since $\mu$ is a convex combination of measures from $M_\sigma(a)$ it also belongs to $M_\sigma(a)$. Therefore, the family $\{M_\sigma(a)\}_{\sigma,a}$ satifies the desired properties. Because the family $\{\mu \mid a \overset{\sigma}{\Longrightarrow}_P \mu\}_{\sigma,a}$ was the smallest possible family indexed with elements $(\sigma,a)\in\Sigma\times A$ satisfying the properties form Example 2.10 we have

$$\{\mu \mid a \overset{\sigma}{\Longrightarrow}_P \mu\} \subseteq M_\sigma(a).$$

Hence, $\mathfrak{s}\alpha \leq \alpha^w$ which completes the proof. □

**Theorem 6.5.** *For simple Segala system coalgebra $\langle A,\alpha\rangle$ and the saturated structures $\mathfrak{s}\alpha$ and $\alpha^w$ from Example 2.10 and Example 3.12 we have*

$$\alpha^w = \mathfrak{s}\alpha.$$

*Proof.* The statement follows directly from Lemma 6.3 and 6.4. □

*E-mail address*: `t.brengos@mini.pw.edu.pl`

Faculty of Mathematics and Information Sciences, Warsaw University of Technology, Koszykowa 75, 00-662 Warszawa, Poland